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ON γ- REGULAR RINGS AND FLATNESS

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ABSTRACT

In this paper we investigate the SF-rings over γ-regular rings and the relations between them. Moreover, we impose some conditions to obtain an interesting property of SSF-rings over γ-regular rings.

1. INTRODUCTION

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Throughout this paper, all rings are associative with identity and all modules are unitary. The concept of Von Neumann regular (briefly, regular) rings was introduced first by J. Neumann in 1936, see [9]. In recent years regularity has been extensively studied by many authors, see [5], [9] and [12]. Rege [11], introduced the idea of simple singular flat rings (briefly, SF-ring), that is if all simple right (left) R-modules are flat, then the ring R is called an SF- ring.

Mahmood and Ibraheem [7] introduced SSF- rings that is, R is said to be right (left) simple singular flat(briefly, SSF-ring), if every simple singular right (left) R- module is flat .A ring R is Von Neumann if and only if every right (left)R-module is flat. It is well known that every regular ring is right and left SF-ring. Ramamurthi, in [15] initiated the study of right (left) SF-rings and of the question whether a right (left) SF-ring is necessarily regular. Since right (left) SF-rings have been extensively studied by many authors and the regularity of right (left)SF-rings which satisfy certain additional conditions is proved [2] ,[11] , [13] and [16], but the question has been remained open. Yue Chi Ming [12], proved the strong regularity of right SF-rings, he also proposed the following question: Is R strongly regular when R right SF-rings? Zhang and Du [2] affirmatively answered the question. Muhammad and Salih[8], introduced the concept of γ-regular rings and showed that every regular ring is γ-regular by add two conditions, also showed the relation between γ-regular rings and strongly regular rings.

In 2015, ZHOW and WEI [17] introduced GWCN ring and gave result showed relation between strongly regular rings and GWCN left SF ring.

In this paper, we find a new characterization between GWCN rings and SF-rings and we investigate the SF-rings over γ-regular rings and the relations between them also we impose some conditions to obtain an interesting properties of SSF-rings over γ -regular rings.

One of the most important rings was introduced by Kandasamy [14], is quasi-commutative rings that is a ring R with $ab = b^m a$ for every $a, b \in R$ when $1 \neq a$ and for some positive integer m. In [8], a condition is given that has a main role in our proofs and is discuss the connection between γ-regular, SF-rings and SSF-rings.

Now we recall the following definitions:

Definition 1.1 [14] Let R be a ring such that for every $a, b \in R$ when m $\neq 1$, there exists a positive integer $m > 1$ such that $ab = b^m a$.

Next, we shall give some basic definitions that are used in this paper:

Definition 1.2 [8] A ring R is said to be γ-regular ring if for every $a \in R$, there exists $x \in R$ and a positive integer $n \neq 1$ such that $a = ax^n a$.

Remark 1.3 [8] Every γ-regular ring is regular ring, however the converse is not true, for example the ring of rational numbers $\mathbb{Q}$ is a field and any field is a regular ring. But the ring of rational

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numbers is not γ-regular ring since $\frac{1}{2} \in \mathbb{Q}$, is regular element but not γ-regular element because can't find $x \in \mathbb{Q}$ such that $a = ax^n a$ for some positive integer $n \neq 1$.

Definition 1.4 [17] A ring R is called generalized weakly central nilpotent **(**briefly, GWCN) if $x^2y^2 = xy^2x$ for all $x \in N(R)$ and $y \in R$.

Definition 1.5 [5] A ring R is called 2-primal if and only if the set of nilpotent elements and the prime radical of R are the same if and only if the prime radical is a completely semiprime ideal.

2. SF- ring and γ-Regular Rings:

In this section we study the effect of SF- rings on γ-regular rings and we discuss some of their properties and gave some main results.

In the first we give the following theorem that is clarify the relation between γ-regular rings and right (left) SF-rings:

Theorem 2.1 Let R be a reduced ring and quasi-commutative. Then R is γ-regular if and only if R is right SF-ring.

Proof: Assume that R is a γ-regular ring, and then is clear that R is a regular ring. Hence R is a right SF-ring.

Conversely, assume that R is a right SF-ring. We need to prove that $aR + r(a) = R$ for any $a \in R$. Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal L containing $aR + r(a)$. But $a \in L$ and R/M is flat, then there exists $B \in L$ such that $a = ba$, Whence $(1 - b) \in L(a) = r(a) \subseteq M$ (because R is reduced), yielding $1 \in M$ which contradicts $L \neq R$. In particular $ar + d = 1$, for some $r \in R$ and $d \in r(a)$, suppose that R is a quasi-commutative ring, then $r^n a + d = 1$ implies $ar^n a + ad = a$, so $a = ar^n a$ for positive integer $n > 1$. Hence R is γ-regular. ■

Theorem 2.2 Let R be a quasi-commutative reduced ring. If R/aR is a right SF- ring, then R is γ - regular.

Proof: Let a be a non-zero element of a ring R and assume that R/aR is a right SF-ring Then for each $a \in a\, R$, there exists $b \in aR$ such that $a = ba$, since $b \in aR$, then $b = ar$, for some $r \in R$. Therefore $a = ba = ara$, since R is a reduced ring and a quasi-commutative ring, then$a = a^2r = a.ar = ar^n a$ for some positive integer $n > 1$.

Hence R is γ-regular. ■

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Following corollary gives a new characterization of γ-regular rings in terms of GWCN rings:

Corollary 2.3 Let R be a quasi-commutative ring. Then R is strongly regular if and only if R is a γ-regular GWCN ring.

Proof: By using similar proof is given in [17, Corollary 2.14] and [8, Theorem 4.4]. ■

Next, we give the following two Lemma [4,5]:

Lemma 2.4 If R is a semi-prime and SRB ring. Then R is reduced.

Lemma 2.5 Let R be a QSRB ring. Then $R/Y(R)$ is a reduced ring.

Theorem 2.6 Let R be a right SF-ring, semi-prime, quasi-commutative and SRB ring. Then R is γ-regular ring.

Proof: By Lemma 2.4, R is a reduced ring. We need to prove that $aR + r(a) = R$, foranya $a \in R$.
` Suppose that $aR + r(a) \neq R$, there exists a maximal right ideal L containing $aR + r(a)$. But $a \in L$ and R/L is a right flat then there exists $b \in L$ such that $a = ba$, whence$(1 - b) \in \ell(a) = r(a) \in L$, yielding$1 \in L$, which contradicts $L \neq R$. In particular, $ar + x = 1$, for some $r \in R$ and $x \in r(a)$, suppose that R is a quasi-commutative ring, then $r^m a + d = 1$ implies $ar^m a + ad = a$, so $a = ar^m a$ for a positive integer $m > 1$. Hence R is γ-regular. ■

Theorem 2.7 Let R be a right SF-ring, quasi-commutative and QSRB-ring. Then R is a γ-regular ring.

Proof: By Lemma 2.5, $R/Y(R)$ is a reduced ring. We claim that $Y(R) = (0)$. Suppose that $Y(R) \neq (0)$, then there exists$0 \neq y \in Y(R)$ such that $y^2 = 0$. Let M be a maximal right ideal containing $r(y)$. Since $r(y)$ is a maximal two-sided ideal of R, then M must be a maximal two-sided ideal of R. On the other hand since R/M is a right flat, and since$y \in M$, there exists $c \in M$ such that $y = cy$, whence $(1 - c) \in r(y) \subseteq M$, yielding $1 \in M$, which contradicts M ≠ R. This proves that R is a reduced ring. We need to show that $a\,R + r(a) = R$, for any $a \in R$.Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal L containing$aR + r(a)$. But $a \in L$ and R/L is a right flat, there exists $b \in L$ such that $a = ba$, whence $(1 - b) \in \ell(a) = r(a) \subseteq L$, yielding $1 \in L$, which contradicts L ≠ R. In particular, $ar + d = 1$, for some $r \in R$ and $d \in r(a)$, and as R is a quasi-commutative ring, then $a = a^2 r = a.ar = ar^m a$ where $m > 1$.

Hence R is γ- regular. ■

According to Rege [11], reduced right SF-rings are strongly regular. Hence we give the following Corollary which is an extension to the Corollary 2.14 in [17]:

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Corollary 2.8 Let R be a quasi-commutative ring. Then R is γ-regular if and only if $V_2(R) = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R\right\}$, is a GWCN ring and R is a right SF- ring.

Proof: By [17, Proposition 2.13] and [8, Theorem 4.4] . ∎

3. SSF-rings and γ-Regular Rings:

First prove the following lemma:

Lemma 3.1 Let R be a quasi-commutative ring and $a \in C(R)$. If $a = ara$ for some $r \in R$, then R is a γ-regular ring.

Proof: Assume that R is a quasi-commutative ring, then for every $a, b \in R$ there exists positive integer $m > 1$ such that $ab = b^m a$. Now, if $a = ara$ for some $r \in R$, so $a = raa$ because $a \in C(R)$, then $a = ar^m a$. Hence R is γ- regular. ∎

Theorem 3.2 Let R be a semicommutative right SSF-ring. Then $C(R)$ is γ-regular.

Proof: First we will show that $aR + r(a) = R$, for any $a \in C(R)$. If not, there exists a maximal right ideal M of R such that $aR + r(a) \subseteq M$.Since $a \in C(R)$, $aR + r(a)$ is an essential right ideal and so M must be an essential right ideal of R. Therefore, R/M is a right flat. So there exists $b \in M$ such that $a = b\,a$. This implies that $(1 - b) \in \ell(a)$. Since R is a semicommutative ring, then $\ell(a) \subseteq r(a)$for every$a \in R$.Therefore, $(1 - b) \in r(a) \subseteq M$ so $1 \in M$, which is a contradiction. Hence $aR + r(a) = R$, for any $a \in C(R)$ and so we have $a = ara$, for some $r \in R$. By Lemma 3.1., we obtain that C(R) is γ-regular. ∎

Lemma 3.3[6] If R is a semicommutative ring. Then $RaR + r(a)$ is an essential right ideal of R, for each $a \in R$.

Theorem 3.4 Let R be a right 2-primal quasi-commutative ring. If R is an SSF-ring, then $R/P(R)$ is γ-regular.

Proof: Let $0' \neq a' \in R' = R/P(R)$.We will show that $R'a'R' + r(a') = R'$. If it is not true, then there exists a maximal right ideal M of R such that$R'a'R' + r(a') \subseteq M/P(R)$.Since R´ is reduced, we have that $r(a') = \ell(a')$, for any $a' \in R'$. Then by Lemma 3.3, $R'a'R' + r(a')$ is an essential right ideal of R'.Then $M/P(R)$must be right essential in R´. Therefore, R/M is simple singular right R-module and R/M is right flat. So, there exists $b' \in M/P(R)$ such that $a' = b'a'$. This implies that $(1 - b') \in \ell(a') = r(a') \subseteq M/P(R)$, this implies that $1 \in M/P(R)$, which is a contradiction. Inparticular $r'a'r' + z = 1$, for some $r' \in R'$ and

$z \in r(a$. Then $r'r'^{m}a' + z = 1$ implies $r'^{m+1}a' + az = a$, so $a = ar'^{m}a$ for a positive integer $m = n + 1 > 1$. Hence R is γ-regular. ■

Lemma 3.5 [7] If $Y(R)$contains no non-zero nilpotent element, then $Y(R) = (0)$.

Lemma 3.6 [3] If R is a reversible ring, then $r(a) = \ell(a)$, for every $a \in R$.

Finally, we give a necessary and sufficient condition for SSF-rings and γ-regular.

Theorem 3.7 Let R be a reversible quasi-commutative ring. Then the following statements are equivalent:

1. R is a γ-regular ring
2. R is a right weakly continuous and a right SSF-ring.

Proof:$(1) \to (2)$ Observe that if R is γ-regular ring then R is regular. It is well known that every SF-ring is SSF-ring. Hence R it a SSF-ring.

$(2) \to (1)$Suppose that $Y(R) \neq (0)$. Then by Lemma 3.5., we may assume that $Y(R)$ is not reduced, So there exists a non-zero $a \in Y(R)$ such that $a^2 = 0$.

We claim that $Y(R) + r(a) = R$. If not, there exists a maximal essential right ideal M containing $Y(R) + r(a)$. Thus R/M is a right flat. Therefore, there exists $b \in M$ such that $a = ba$. This implies that $(1 - b) \in \ell(a)$. Since R is reversible, then by Lemma 3.6,$\ell(a) = r(a)$, so $(1 - b) \in r(a) \subseteq M$.Thus, $1 \in M$, which is a contradiction . Therefore, $Y(R) + r(a) = R$. Hence we can write $1 = c + d$, for some $c \in Y(R)$ and $d \in r(a)$.Thus, $a = ca$ and so $(1 - c)a = 0$.Since $c \in Y(R) = J(R)$, $(1 - c)$ is invertible. Thus $a = 0$, which is also a contradiction. Therefore, $Y(R)$ is reduced and $Y(R) = (0)$. Hence R is regular, then for every $a \in R$. Suppose that R is a quasi-commutative ring, and then there exist $b \in R$ such that $a = aba = a^2b = a.ab = ab^n a$ for some positive $n > 1$ then R is a γ-regular ring. ■

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پوختە

لەم توێژینە وەیەدا، ئێمە لێکوڵینەوەمان لە بازنەی (SF) لە جۆری بازنە ی ڕێکخراوی ئاسایی (γ) و ئە و پەیوەندیانەی لە نێوانیان دا هەن کردووە. سەرەڕای ئەمە هەندێک مەرج دەخەینە پاڵ بۆ بەدەست هێنان و گەیشتن بە تایبەتمەندی باڵکێش لە بازنەکانی (SSF) لە جۆری بازنەی ڕێکخراوی (γ) دا.

مستخلص

فى هذا البحث نبحث عن الحلقات SF بالنسبه للحلقات المنتظمه γ والعلاقه بينهم . بالاضافه الى ذلک , نقوم باضافه بعض الشروط للحصول على الخواص المهمه للحلقات SSF بالنسبه للحلقات γ .