QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue (3), Summer 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

On Strong γ- Regularity and SPF-Rings

Dina Ameer Basheer hadid

Dept. of Mathematics, , University of Lebanese French, Kurdistan region, Iraq.

***E-mail**: dena.hadid@yahoo.com*

.

ARTICLE INFO

Article History:
Received: 11/5/2018
Accepted: 9/6/2018
Published: Summer 2018

Doi:
10.25212/lfu.qzj.3.3.28

Keywords:

strongly γ-regular ring, SF-rings, p – injective, SPF-rings.

ABSTRACT

In this paper we study a definition of SF – rings over strongly γ - regular rings and we find some properties and main results for it by adding some conditions. Moreover, we continue to study SPF-rings over strongly γ - regular ring and we discuss several properties for it..

1. Introduction

Throughout this paper, every ring is an associative with unity. A ring R is said to be Von Neumann regular (regular) [4], if for every $a \in R$, there exists $x \in R$ such that $a = axa$, Consequently, R is called strongly regular ring if for every $a \in R$,there exists $x \in R$ such that $a = a^2x$.

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue (3), Summer 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

Muhammad and Salih[10], called a ring R is γ-regular ring if for every $a \in R$,there exists $x \in R$ and a positive integer $n \neq 1$ such that $a = ax^n a$, Consequently, R is called strongly γ-regular ring if for every $a \in R$,there exists $x \in R$ and a positive integer $n \neq 1$ such that $a = a^2 x^n$.It should be noted that in a strongly γ-regular ring R, $a = a^2 x^n$ if and only if $a = x^n a^2$.Since if $n = 1$ then for every $a \in R$ there exists $x \in R$ such that $a = ax^1 a$. Rege [11], study SF- rings and called R/I a flat right (left) R-module. He proved that any SF-ring is regular.

Mahmoud and Ibrahim [6], called a ring R/I is singular if I . Also called a ring R is a right (left) simple singular flat (SSF-ring) if every simple singular right (left) R-module is flat. Recall that R is reduced if it has no non-zero nilpotent element, R is semi-prime ring if it contains no non-zero nilpotent ideals [4]. A ring R is said to be zero commutative insertive (briefly ZC) if for , $b \in R$, $ab = 0$ implies $ba = 0$ [5] . A ring R is called ERT (resp. MERT) if every essential (resp.–maximal essential, if it exists) right ideal of R a two- sided ideal. Similarly, ELT (resp. MELT) rings are defined on using left ideal [6].

A right *R*-module *M* is *P*-injective if, for any principal right ideal *P* of *R*, any right *R*-homomorphism *g: p*⍰ *M*, there exists $y \in M$ such that $g(b) = yb$ for all $b \in p$. For a subset I of R. A ring R is called a right SPF-ring if every simple right R–module is either P- injective or flat [9].

The left annihilator of I in R is $\ell(I) = \{r \in R : rx = 0,\text{for all } x \in R\}$, likewise for the right annihilator $r(X)$. $Y(R)$ and $J(R)$ will stand respectively, for the right singular ideal and the Jacobson radical of R [4].

The main goal of the work is to study a strongly γ -regular rings, which was introduce by Mohammad A. J. and Salih. S. M. in (2006)[10]. That is, a ring R is said to be strongly γ -regular if for every $a \in R$ there exists $b \in R$ and a positive integer $n \neq 1$ such that $a = a^2 b^2$. We will study some basic properties of those rings. Finally, we show the relation between strongly γ -regular rings and other rings

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue ( ), 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

2. Some Basic Definitions:

In this section we view some basic definitions and relations on γ-regular rings.

Definition 2.1 [10]

A ring R is said to be a quasi-commutative if for every $a, b \in R$ when $1 \neq a$ there exists m positive integer such that $ab = b^m a$.

Remark 2.2 [10]

(i) For every $a, b \in R$ when $1 \neq a$ there exists $m > 1$ positive integer such that $ab = b^m a$, For $a = 1$ the above condition does not satisfy.

(ii) For $1 \cdot b = b^m \cdot 1$ then $b = b^m$ and this is a trivial case where $m = 1$.

Lemma 2.3 [**11**]

Let I be a right (left) ideal of R. Then R/I is a flat right (left) R-module if and only if for each $a \in I$, there exists $b \in I$ such that $a = ba(a = a\, b)$.

Definition 2.4 [11]
A ring R is called right (left) simple flat (briefly, SF-ring) if every simple right (left) R-module is flat.
Definition 2.5 [6]

Let R be a ring, and I an ideal of R, then R/I is called singular if I is an essential right ideal of R.

Definition 2.6 [11]
A ring R is said to be quasi-strongly right bounded (briefly, QSRB) if every non-zero maximal right ideal contains a non-zero two-sided sub ideal of R.
Lemma 2.7 [11]

Let R be a QSRB-ring. Then $R/Y(R)$ is a reduced ring.

Definition 2.8 [1]

A ring R is called reversible if $ab = 0$ implies $ba = 0$ for every $a, b \in R$.

Proposition 2.9 [3]

(1) Left or right WPZI rings are abelian.
(2) Left or right PZI rings are abelian.

Corollary 2.10 [3]

The following conditions are equivalent for a ring R :

(1) R is a reduced ring.
(2) R is a semicommutative ring and left NV rings.

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue ( ), 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

Theorem 2.11 [7]

Let *R* be a *ZC* ring and let M be a maximal right ideal *M* of *R* such that r*(a)*� *M* for all a�*M*. Then *M* is an essential right ideal of *R.*

Proposition 2.12 [9]

If R is a right SF-ring, then any left regular element is right invertible in R.

3. Strong γ-Regularity of SF-ring

Rege[11], pointed out that if R is a reduced right(left) SF-rings, then R is a strongly regular ring. We can extend this result to strongly γ-regular rings.

Proposition 3.1

Let R be a quasi-commutative. If R is a right SF- ring and right PZI ring, then R is a strongly γ -regular ring.

Proof.

Assume that a ∈ R. If a = 0, we are done. If a ≠ 0, then there exists n ≥ 1 such that $a^n \neq 0$ and $r(a^n)$ is an ideal of R because R is a right PZI ring. If aR + r(a^n) ≠ R, then there exists a maximal right ideal M of R containing aR + r(a^n). Since R is a right SF -ring, R/M is a flat right R-module, so there exists b ∈ M such that a = ab because a ∈ M . Hence 1 − b ∈ r(a^n) ∈ M , which implies 1 ∈ M , a contradiction. Therefore aR + r(a^n) = R. Let 1 = ac + x, where c ∈ R and x ∈ r(a^n). So $a^n = a^n ac$, write b = a^{n-1}− a^{n-1}ac. Then b^2= 0. If b ≠0, then similar to the proof mentioned above, we have bR + r(b) = R, so there exists d ∈ R such that b = bdb. Hence there exists x ∈ R such that a^{n-1} = a^{n-1}xa. If b = 0, then a^{n-1} = a^{n-1}ca. Repeating the process above, we can obtain that a = awa for some w ∈ R. By Proposition 2.9, R is an Abelian ring, so R is strongly regular ring. Since R is quasi-commutative ring as in Remark 2.2 , then for every $a, b \in R$, $ab = b^n a$ for some positive integer $n > 1$. Then $a = ab^n a$, and since R is strongly regular , hence R is reduced , implies $a = a^2 b^n$. Therefore R is strongly γ-regular rings .

Because semicommutative rings are right PZI with quasi-commutative ring as in Remark 2.3, we have the following corollary:

Corollary 3.2

Let R be a quasi-commutative ring and R is a right weakly semicommutative right SF- ring. Then R is strongly γ-regular. ■

Since reversible rings are semicommutative by C**orollary 2.10,** we obtain the following corollary:

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue ( ), 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

Corollary 3.3

Let R be a quasi-commutative ring. If R is a right SF - ring and reversible ring, then R is a strongly γ - regular ring. ■

Theorem 3.4

Let R be a quasi-commutative ring and reduced ring such that for every $a \in R$, $R/r(a^n)$ is an SF-ring, for some positive integer n. Then R is strongly γ-regular ring.

Proof.

Let a be a non-zero element in R and $a'^n = a + r(a^n) \in R/(a^n)$, for some positive integer n. Clearly, $a'^n \neq 0'$ because otherwise if $a'^n = 0'$, then $a + r(a^n) = r(a^n)$ implies $a \in r(a^n)$ and hence $a^{n+1} = 0$ gives $a = 0$, since R is reduced, this is contradiction .Let $a'^n x' = 0'$, now prove that $x' = 0'$.

Observe that $(a + r(a^n))(x + r(a^n)) = r(a^n)$, implies $ax + r(a^n) = r(a^n)$, and hence $a\,x \in r(a^n)$, so $(a^{n+1})x = 0$. Thus $x \in r(a^{n+1})$.Since R is reduced, therefore $x \in r(a^n)$ implies $a^n x = 0$, whence $x \in r(a^n)$. Hence $x' = 0'$, this means that a'^n is a right non-zero divisor. Likewise we can prove that a'^n is a left non-zero divisor.

Since $R/r(a^n)$ is SF-ring, then by Proposition 2.12, a'^n is invertible element. Then there exists $0' \neq y'^n = y + r(a^n) \in R/r(a^n)$ such that $a'^n y' = 1$. Then $(a + r(a^n))(y + r(a^n)) = 1 + r(a^n)$. So $(a\,y - 1) \in r(a^n)$, and $a^n(ay - 1) = 0$.Thus $a^n = a^{n+1}y = a^n a\,y = a^n y^m$ where $m > 1$ because R is a quasi-commutative ring as defined in Remark 2.2. So $(1 - y^m) \in r(a^n)$, since R is reduced, then $r(a^n) = r(a)$ and $r(a) = l(a)$, whence $(1 - y^m a)a = 0$. Then $a = a^2 y^m = y^m a^2$ for $m > 1$.

Now $a = y^m a^2$, then $a\,y^m = (y^m a^2)y^m = y^m(a^2 y^m) = y^m\,a$ this implies $ay^m a = y^m a^2 = a$, then $a = ay^m a$.. Since R is quasi-commutative ring as in Remark 2.1, then for every $a, b \in R$, $ay = y^n a$ for some positive integer $m > 1$ and R is reduced, hence $a = a^2 y^m$. Therefore R is strongly γ-regular rings. ■

Theorem 3.5

Let R be a quasi-commutative ring .Then the following are equivalent:

1. R is strongly γ-regular rings.
2. R is QSRB and right SF-ring.

Proof: $(1) \rightarrow (2)$ is obvious

$(2) \rightarrow (1)$, by Lemma 2.7, R/Y is a reduced ring.Suppose that $Y = 0$, if $Y \neq 0$ then there exists $0 \neq y \in Y$ suh that $y^2 = 0$.Let M be a maximal right ideal containing $r(y)$. Since $r(y)$ is an essential two-sided ideal of R, then M must be an essential two-sided ideal of R. Since R/M is flat module, and since $y \in Y$, there exists $x \in M$ such that $= yx$, then $(1 - x) \in r(y) \subseteq M$ yielding $1 \in M$ which contradicts $M \neq R$.This proves that R is a reduced ring.

In order to show that R is strongly γ-regular rings we need to prove that $aR + r(a) = R$ for any $a \in R$.Suppose that $aR + r(a) \neq R$, then there exists a maximal right ideal L containing $aR + r(a)$. But $a \in L$ and R/M is flat, there exists $B \in L$ such that $a = ba$, whence $1 - b \in L(a) = r(a) \subseteq M$ (because R is reduced), implies $1 \in M$ which contradicts $L \neq R$. In particular $ar + p = 1$, for some $r \in R$ and $p \in r(a)$.Since R is a quasi-commutative ring as defined in Remark 2.2 and reduced. Then $a = a^2 r^n$ for positive integer $n > 1$ hence R is strongly γ-regular. ■

Theorem 3.6

Let R be a quasi-commutative ring and right SF-ring with every nilpotent element of R is central. Then R is strongly γ-regular.

Proof.

Let a be a non-zero element in R with $a^2 = 0$, and let M be a maximal right ideal containing $r(a)$.Since $a \in r(a) \subseteq M$, and since R/M is flat, there exists $b \in M$ such that $a = ba$, this implies that $(1-b) \in l(a)$, but every nilpotent is central gives $r(a) = l(a)$, hence $(1-b) \in r(a) \subseteq M$, yielding $1 \in M$, and this contradicts $M \neq R$. Therefore, R is a reduced ring. By a similar method of proof used in Theorem 3.5 , R is strongly γ-regular. ■

4. Strongly γ-regular ring with SPF- Rings:

In this section, we study the connection between SPF- rings and strongly γ -regular rings.

If R satisfies a polynomial identity with coefficients in the centroid and least one coefficient is invertible, then R is called P.I – ring [7].

Theorem 4.1

Let R be a quasi-commutative ring ERT, ZC and SPF-ring. Then R is strongly γ -regular.

Proof.

Let a∈R. If aR + r(a)≠ R, then there exists a maximal right ideal M containing aR + r(a). Since a ∈ M and r(a) ⊆ M, then by Lemma 2.11 , M is essential and two-sided. If R/M is flat, then a = ca for some c ∈M and (1-c) ∈r(a) ⊆M , yielding 1∈M , a contradiction . If R/M is P-injective, the right R-homomorphism g : aR ∈ R/M defined by $g(ab) = b + M$ for all b ∈ R yielding 1 + M = g(a) = da + M for some d∈R, Therefore 1∈M , again a contradiction, Thus $\mathrm{R} = \mathrm{aR} + \mathrm{r(a)}$. In particular $1 = \mathrm{ax} + \mathrm{y}$, $\mathrm{y} \in \mathrm{r(a)}$, x ∈ R . Thus $a = a^2x$, then R is strongly regular rings. Since R is quasi-commutative ring as defined in Remark 2.2, then R is reduced, hence a $= a^2x^n$. Therefore R is strongly γ -regular.

Theorem 4.2

Let R be a quasi-commutative. If R is a semi-prime, MELT, SPF, P. I – ring, then R is strongly γ – regular.

Proof.

It is sufficient to prove that any left ideal I of R that is idempotent. The result then follows from [2].For any t ∈ I, if $T = RtR$, then $T + r(T)$ is essential in R . Since R is semi-prime, $r(T) = L(T) \subseteq L(t)$, and if $T + L(t) \neq R$, let L be a maximal left ideal containing T+L(t), and let M be a maximal right ideal containing L. Then If R/M is flat which implies $t = dt$ for some $d \in M$ and $(1-d) \subseteq L(t) \in M$ implies 1∈ M, a contradiction, and if R/M is P-injective , then any right R–homomorphism of tR into R/M extends to one of R into R/M. let f:tR→R/M be defined by $f(tR) = R + M$. Note that f is a well-defined R– homomorphism. Indeed, let $r_1, r_2 \in R$ such that $tr_1 = tr_2$. Then $tr_1 - tr_2$ = 0 , implies that ($r_1 - r_2$)= 0, so $r_1 - r_2 \in r(t) \subseteq l(t) \in M$, . Hence, $r_1 + M = r_2 + M$. Now R/M is P- injective. There exists $a \in R$ such that 1+$M = f(t) = at + M$, whence $1 \in M$, again a contradiction. Thus $T + L(t) = R$ and $t \in RtRt$ which proves $I = I^2$, then for every ∈ R , there exists $b \in R$ such that $a = aba$. Since R quasi-commutative ring as defined in

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue ( ), 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

Remark 2.2, then R is reduced. Therefore R is strongly γ − regular.

We consider other conditions for a right SPF ring to be strongly γ –regular.

Theorem 4.3

Let R be a quasi-commutative right SPF–ring and ZC - ring with every nilpotent element of R is central. Then R is strongly γ -regular.

Proof.

Let a be a non-zero element in R with $a^2 = 0$, and let M be a maximal right ideal containing r(a).Since $a \in r(a) \subseteq M$, and since R/M is flat, there exists $b \in M$ such that $a = ba$. This implies $1 - b \in l(a)$. Since every nilpotent is central we obtain $r(a) = l(a)$. Whence $1 - b \in r(a) \subseteq M$, yielding 1∈ M, which contradiction $M \neq R$. Now, if R/M is P-injective, the right R–homomorphism $g: aR \to R/M$ defined by $g(ab) = b + M$ for all $\in R$, yields $1 + M = g(a) = ca + M$ for some $c \in R$, hence 1∈ M a gain contradiction. Therefore R is a reduced ring. For any *b⍰R*, set *L=RbR+r(b)*. Since *RbR⍰r(b)=RbR⍰r(RbR)=0*, (because R is reduced), then *L = RbR ⍰ r (b)*. Suppose that *L ⍰ R.* If M is a maximal right ideal of *R* containing *L*, first suppose that *R/M* is *P*-injective. The right *R*-homomorphism *g:bR⍰R/M* defined by *g(ba)=a+M* for all *a⍰R* yields *1+M=g(b)=cb+M* for some *c⍰R*. Then *1⍰M,* contradicting *M⍰R* .If *R/ M* is flat, then *b= d b* for some *d⍰M* , implies that *1⍰M*, again a contradiction. Thus *R = L* which proves that *RbR=eR*, *e=e²⍰R*. *T*herefore *R* is biregular. then there exist $x \in R$ such that $xbx = e.x$, *e=e²⍰R*. Since R is quasi-commutative ring as defined in Remark 2.2 , then $x^2 b^m = x$ for positive integer $m > 1$. Then R is strongly γ -regular.

Theorem 4.4

Let R be a quasi-commutative .If R is an SPF–ring such that for any $a \in R$, there exists $b \in R$ such that $RaR = r(b) = l(b)$, then R is strongly γ -regular .

Proof.

If $RaR = r(b) = l\,(b)$ and $+l(b) \neq R$, let M be a maximal right ideal containing $bR + l(b)$. If R/M is flat, then $b = db$ for some $d \in M$ and $1 - d \in l(b) \subseteq M$ implies $1 \in M$, a contradiction. If R/M is P-injective, then the right R–homomorphism $f: bR \to R/M$ defined by $f(bc) = c + M$ for all $c \in R$, yields $1 + M = f(b) = db + M$ for some $d \in R$, whence $1 \in M$, again a contradiction. Thus $bR + l(b) = R$ and $b = bcb$, for $c \in R$. Then $RaR = r(e) = (1 - e)R$, where $e = cb$ is idempotent. By a similar method of proof as used in **Theorem 4.3** , R is strongly γ -regular.

REFERENCES

[1] Cohn P. M., *"Reversible rings"*, Bull. London Math. Soc., 31(1), (1999), pp.641-648.
[2] Fisher J.W., and Akmendaviz E.P., (1973), *" Regular P.I-rings ", Pro. of the Amer. Math. Society*, **39**:247-252 .
[3] JUNCHAO W. N., (2011). *"STRONG REGULARITY OF SF-RINGS",* ACTA MATH. VIETNAMICA, Volume 36 , Number 3 , 677-683 .
[4] Lam T. Y., "***Exercises in Classical Ring Theory***", Springer-Verlag, New York Berlin Heidelberg, (2000).
[5] Liang L. Wang L. and Liu Z. (1999), *"On a generalization of semicommutative rings"*, Taiwanese J. of Math., 11(5), pp.1359-1368.

QALAAI ZANIST SCIENTIFIC JOURNAL
A Scientific Quarterly Refereed Journal Issued by Lebanese French University – Erbil, Kurdistan, Iraq
Vol. (3), Issue ( ), 2018
ISSN 2518-6566 (Online) - ISSN 2518-6558 (Print)

[6] Mahmood R. D. and Abdul-Jabbar A. M., *"On rings whose simple singular R-modules are flat, I"*, Raf. J. of Comp. Sc. and Math., 7(1), (2010), pp.51-57.
[7] Mahmood R. D. and Ibraheem Z. M. (2007) , *" ON REGULARITY AND SPF-RINGS"*, J.J. APP. SCI. VOL. 9, NO. 1, 1-6 .
[8] Ming R.Y.C., (1995), *"A note on regular rings"*, II", *Bull. Math. Soc. Sc. Math. Roumanie*, **38** : 3-4
[9] Ming R. Y. C., *"On injectivity and P-injectivity IV "*, Bull. Korean Math. Soc. 40(2), (2003), pp.223-234.
[10] Mohammad A. J. and Salih S. M." *On γ -regular rings*", J. Edu. Sci., 18(4), (2006), pp.118-139.
[11] Rege M. B., *"On Von Neumann regular rings and SF-rings"*, Math. Japonica, 31(6), (1986), pp.927-936.